Water flow in soils (I)

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Today's Lecture

- **Movement of water in soils**
- **Darcy's law and conservation of mass of water in soils**
- **Flow in saturated soils**
- **Hydraulic conductivity measurements** (laboratory and field)

See Notes 3.pdf (up to page 14)

Transport phenomena

Transport phenomena (e.g., heat and mass transfer) generally involve two types of processes:

- movement, as described by an **equation of motion**;
- temporal variations of the stock resulting from:
  - external factors (precipitation, evaporation, etc.)
  - local consumption or production (root extraction, biological processes, etc.)
  - exchanges between phases (frost, condensation, evaporation, etc.)

Stock variations are described by the conservation of mass or energy (**continuity equation**)

Laminar flow in narrow tubes

Adjacent layers in a flowing fluid transmit tangential stresses (drag) due to the attraction between fluid molecules. The relationship between the drag force and flow velocity is known as Newton's law of viscosity:

$$\tau = \frac{F}{A} = \eta \frac{dv}{dy} \quad \longrightarrow \quad \frac{dv}{dy} = \frac{\tau}{\eta}$$

where τ is the shearing stress (a force *F* acting on an area *A*), η is the coefficient of viscosity [Pa s], v is fluid velocity, and *y* is a spatial coordinate perpendicular to the direction of flow.

Now consider a fluid flowing through a cylindrical tube having diameter 2R. Assuming laminar flow (caused by the pressure gradient ΔP). The pressure force on a fluid cylinder of length L and radius y is $\Delta P \pi y^2$, which must be equal to the frictional resistance force, $2\pi y L \tau$, acting on the circumference of the fluid cylinder. We thus obtain:

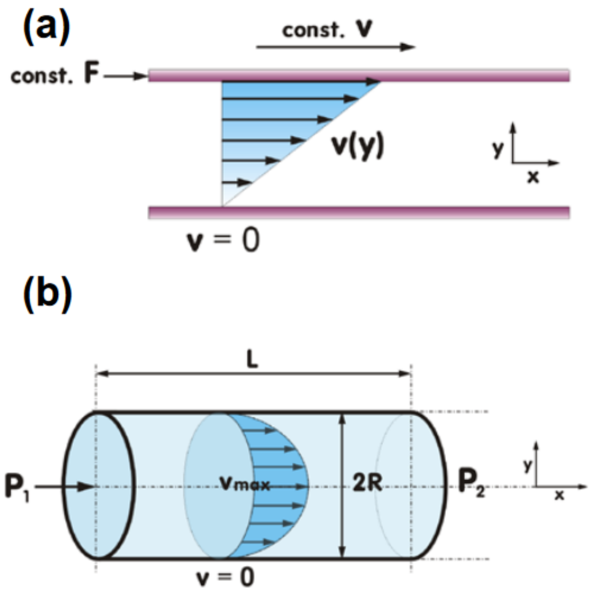

Fig.2-1: Velocity Distributions (a) in a viscous fluid between parallel plates, and (b) in a cylindrical tube

Source: Or, Tuller, & Wraith, 1994-2018

$$\frac{dv}{dy} = -\frac{\Delta P y}{2\eta L} \quad \longrightarrow \quad \int_{v(y)}^{v(R)=0} dv = -\frac{\Delta P}{2\eta L}\int_{y}^{R} y \cdot dy \quad \longrightarrow \quad \boxed{v(y) = \frac{\Delta P}{4\eta L}(R^2 - y^2)}$$

Note: the negative here sign arises from the decrease in v with the increase in y from the center

Laminar flow in narrow tubes

The expression for $v(y)$ describes a parabolic velocity distribution, with the maximal velocity being along the central axis (i.e., y=0):

$$v_{max} = \frac{\Delta P}{4\eta L} R^2$$

To calculate the volume (quantity) of water flowing through the tube per unit time, Q [L^3/t], we need to integrate the velocity over the cross-sectional area of the tube (note, we use a cylindrical coordinate system):

$$Q = \int_0^{2\pi}\int_0^{R} v(r)\cdot \underbrace{r\cdot dr\, d\vartheta}_{dA = dxdy} = 2\pi \frac{\Delta P}{4\eta L}\int_0^R (R^2 - r^2) r \cdot dr = \frac{\pi \Delta P}{2\eta L}\left(\frac{R^4}{2} - \frac{R^4}{4}\right)$$

Average velocity:

Poiseuille's Law: $$Q = \frac{\pi \Delta P R^4}{8\eta L} \longrightarrow \bar{v} = \frac{Q}{\pi R^2} = \frac{R^2}{8\eta}\left(\frac{\Delta P}{L}\right)$$

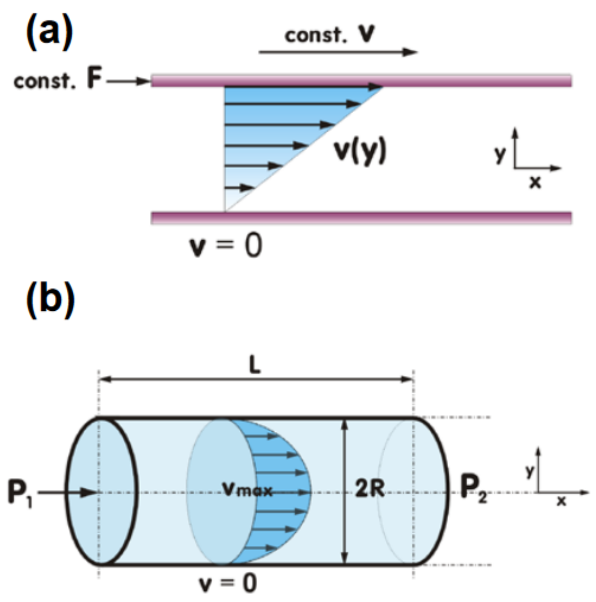

Fig.2-1: Velocity Distributions (a) in a viscous fluid between parallel plates, and (b) in a cylindrical tube

Source: Or, Tuller, & Wraith, 1994-2018

This shows that the volume of flow is proportional to the pressure drop per unit distance (ΔP/L) and the fourth power of the radius of the tube. Poiseuille's Law is often used as a model for flow in soil pores

Laminar flow in narrow tubes

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Example 2-1: Laminar Flow in Tubes

Problem Statement:

What is the average (laminar) flow velocity of water at 20°C through a 50 m long tube having diameter d=0.1 m under a pressure difference of 100 Pa?

Solution:1

We introduce into Eq.(50) L=50 m, ΔP=100 Pa, $R^2=(0.1/2)^2$ m^2, and the viscosity of water at 20°C (taken from tabular values) is 0.001 Pa s. The resulting mean velocity is:

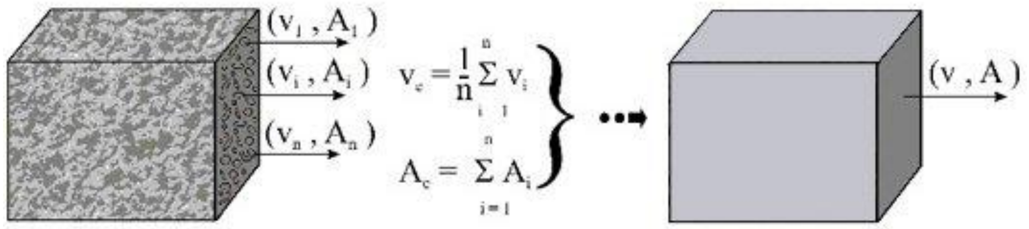
$$\overline{v} = \frac{0.0025\,\mathrm{m}^2}{8 \times 0.001\,\mathrm{Pa\,s}}\left(\frac{100\,\mathrm{Pa}}{50\,\mathrm{m}}\right) = 0.625\,\mathrm{m/s}$$

Source: Or, Tuller, & Wraith, 1994-2018

Self-Study

Darcy’s Law

Soil pores do not resemble uniform and smooth tubes, which form the basis for Poiseuille's Law. In most cases soil pores are highly irregular, having an intricate geometry which prohibits microscopic description of flow pathways. For this reason, flow in soils and other porous media is generally described using macroscopic or averaging terms. In this type of representation, the detailed flow pattern is replaced by an equivalent average of the microscopic velocities crossing a control plane in the porous medium.

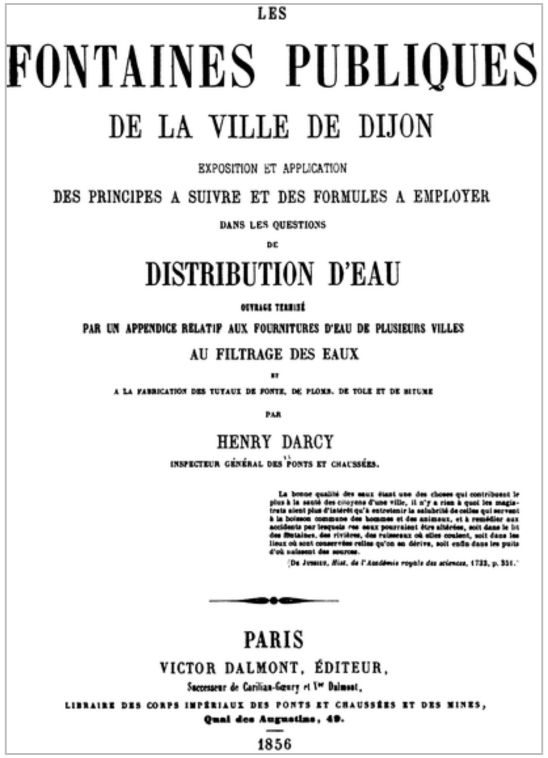


Source: https://echo2.epfl.ch/VICAIRE/mod_3/chapt_5/main.htm

Darcy's Law

The first quantitative description of flow through a porous medium was reported by the French engineer, **Henri Darcy** (1856), who was in charge of enlarging and modernizing the water works of Dijon.

Sand filters were used at the time, but the physics of the water flow through porous media was completely unknown so Darcy conducted experiments on water filtration (using a 3.5 m vertical tank).



LES

FONTAINES PUBLIQUES

DE LA VILLE DE DIJON

EXPOSITION ET APPLICATION

DES PRINCIPES A SUIVRE ET DES FORMULES A EMPLOYER

DANS LES QUESTIONS

DE

DISTRIBUTION D'EAU

OUVRAGE TERMINÉ

PAR UN APPENDICE RELATIF AUX FOURNITURES D'EAU DE PLUSIEURS VILLES

AU FILTRAGE DES EAUX

ET

A LA FABRICATION DES TUYAUX DE FONTE, DE PLOMB, DE TOLE ET DE BITUME

PAR

HENRY DARCY

INSPECTEUR GÉNÉRAL DES PONTS ET CHAUSSÉES.

La bonne qualité des eaux étant une des choses qui contribuent le plus à la santé des citoyens d'une ville, il n'y a rien à quoi les magistrats aient plus d'intérêt qu'à entretenir la salubrité de celles qui servent à la boisson commune des hommes et des animaux, et à remédier aux accidents par lesquels ces eaux pourraient être altérées, soit dans le lit des fontaines, des rivières, des ruisseaux où elles coulent, soit dans les lieux où sont conservées celles qu'on en dérive, soit enfin dans les puits d'où naissent des sources.

(De Jussieu, *Hist. de l'Académie royale des sciences*, 1733, p. 351.)

PARIS

VICTOR DALMONT, ÉDITEUR,

Successeur de Carilian-Gœury et Vve Dalmont,

LIBRAIRE DES CORPS IMPÉRIAUX DES PONTS ET CHAUSSÉES ET DES MINES,

Quai des Augustins, 49.

1856

Wikipedia

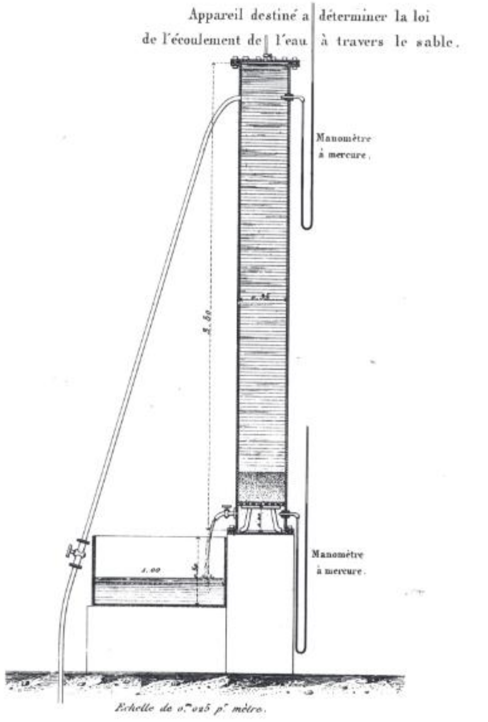


Fig. 8.21 Drawing of the original experimental set-up as presented by Darcy (1856).

Source: Brutsaert (2005)

Darcy's Law

The findings of Darcy can be summarized by the following equation:

$$q = \frac{Q}{A} = K\frac{\Delta H}{L}$$ Darcy's Law

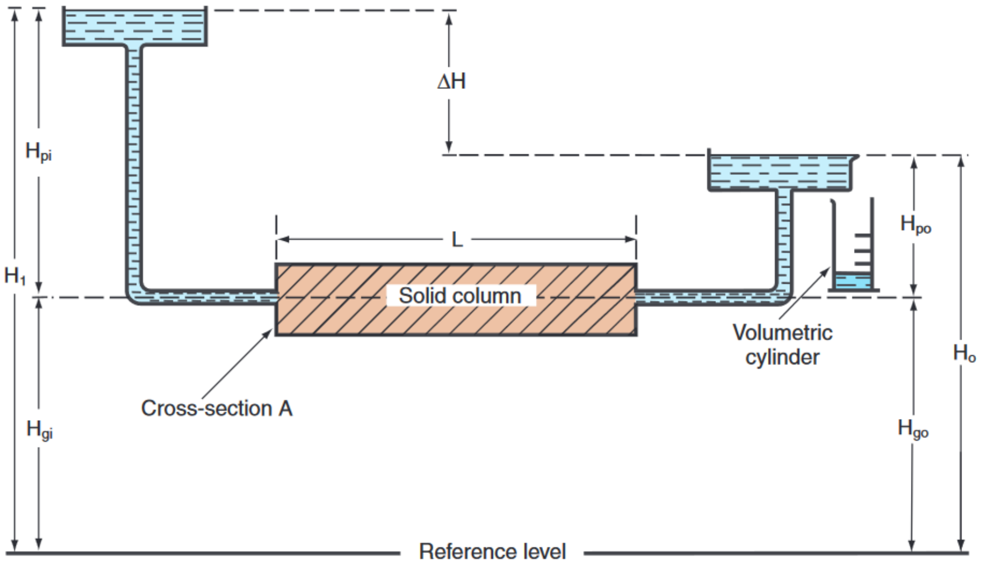


Fig. 7.3. Flow in a horizontal saturated column. Hillel (2003)

where *q* is the **water flux** density (the discharge rate Q=Volume/time flowing through a cross-sectional area A), *K* is a proportionality constant known as the **saturated hydraulic conductivity**, and ΔH is the difference in **hydraulic potential** between two points separated by a distance L.

Because the hydraulic head *H* does not vary linearly along the streamlines, local values of the hydraulic slope must be considered. At the limit, when *L→0*, the finite difference ratio *ΔH/L* is replaced by *dH/dx* , and Darcy's law in the differential form is:

$$q = -K\frac{dH}{dx}$$

Note: The negative sign is introduced as flow occurs in a direction of decreasing potentials

Flow in saturated soils (equilibrium)

Gravitational, pressure, and total hydraulic head

The water pressure is notequal along the column, being greater at the bottom than at the top of the column.

Why, then, will the water not flow from a zone of higher pressure to one of lower pressure?

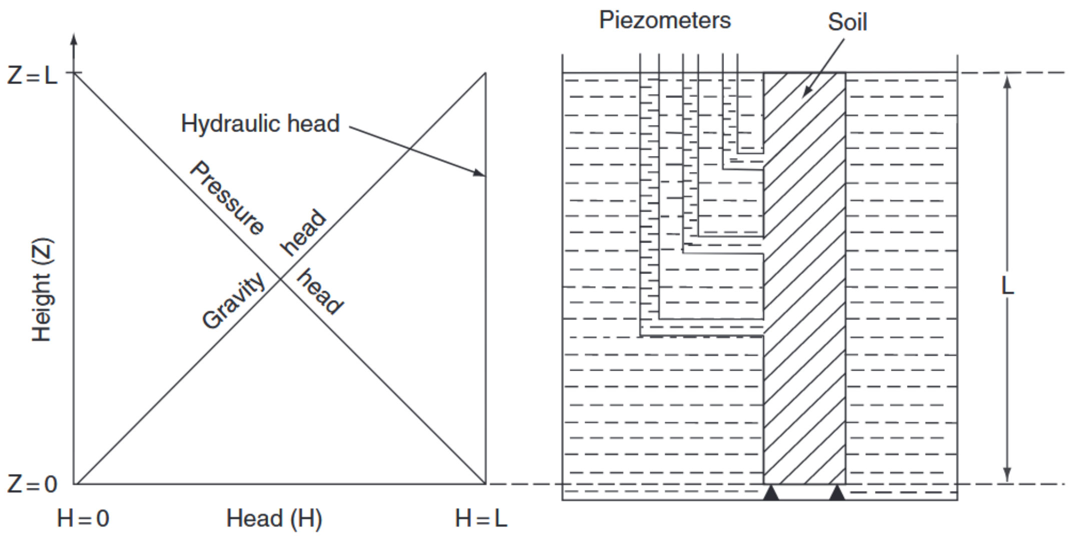


Fig. 7.4. Distribution of pressure, gravity, and total hydraulic heads in a vertical column immersed in water, at equilibrium.

Hillel (2003)

Two opposing gradients exists (pressure and gravitational) which in effect cancel each other, so the total hydraulic head is constant throughout the column (indicated by the piezometers).

Remember: H = h + z $\Delta H = 0$ ⟶ No flow

Flow in saturated soils (horizontal column)

Water flow in a horizontal column occurs in response to a pressure head gradient. Flow in a vertical column may be caused by gravitation as well as pressure. The *gravitational head* H_g at any point is determined by the height of the point relative to some reference plane, while the pressure head is determined by the height of the water column resting on that point. The total hydraulic head H is the sum of these two heads:

$$H = H_p + H_g \tag{7.8}$$

To apply Darcy's law to vertical flow, we must consider the total hydraulic head at the inflow and at the outflow boundaries (H_I and H_o, respectively):

$$H_i = H_{pi} + H_{gi} \qquad \text{and} \qquad H_o = H_{po} + H_{go}$$

Darcy's law thus becomes

$$q = K[(H_{pi} + H_{gi}) - (H_{po} + H_{go})]/L$$

The gravitational head is often designated as z, which is the vertical distance in the rectangular coordinate system x, y, z. It is convenient to set the reference level as the point $z = 0$ at the bottom of a vertical column or at the center of a horizontal column. However, the exact elevation of this hypothetical level is unimportant, since the absolute values of the hydraulic heads determined in reference to it are immaterial and only their differences from one point in the soil to another affect flow.

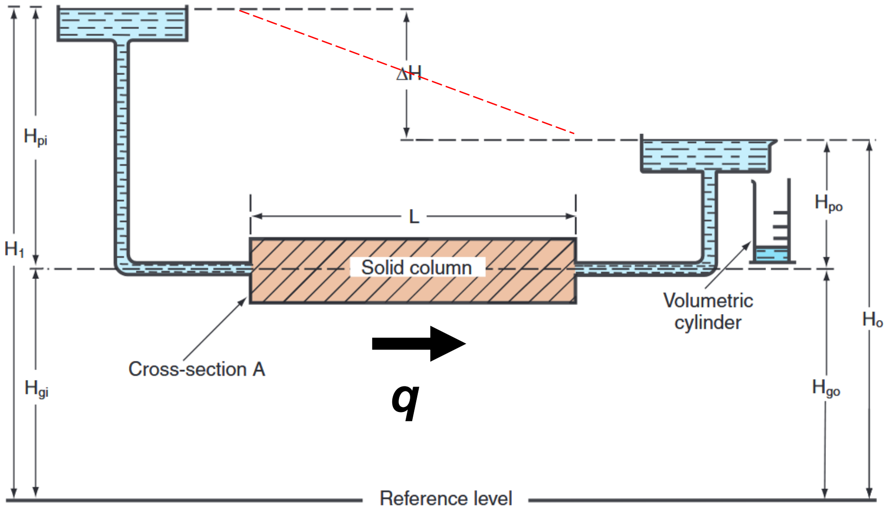


Fig. 7.3. Flow in a horizontal saturated column.

Hillel (2003)

Flow in saturated soils (vertical column)

In order to calculate the flux according to Darcy's law, we must know the hydraulic head gradient, which is the ratio of the hydraulic head drop (between the inflow and outflow boundaries) to the column length, as shown here:

		Pressure head		Gravity head
Hydraulic head at inflow boundary H_i	=	H_1	+	L
Hydraulic head at outflow boundary H_o	=	0	+	0
Hydraulic head difference $\Delta H = H_i - H_o$	=	H_1	+	L

The Darcy equation for this case is

$$q = K\,\Delta H/L = K(H_1 + L)/L = KH_1/L + K \tag{7.9}$$

Comparison of this case with the horizontal one shows that the rate of downward flow of water in a vertical column is greater than in a horizontal column by the magnitude of the hydraulic conductivity. It is also apparent that, if the ponding depth H_1 is negligible, the flux is equal to the hydraulic conductivity. This is due to the fact that, in the absence of a pressure gradient, the only driving force is the gravitational head gradient, which, in a vertical column, has the value of unity (since this head varies with height at the ratio of 1:1).

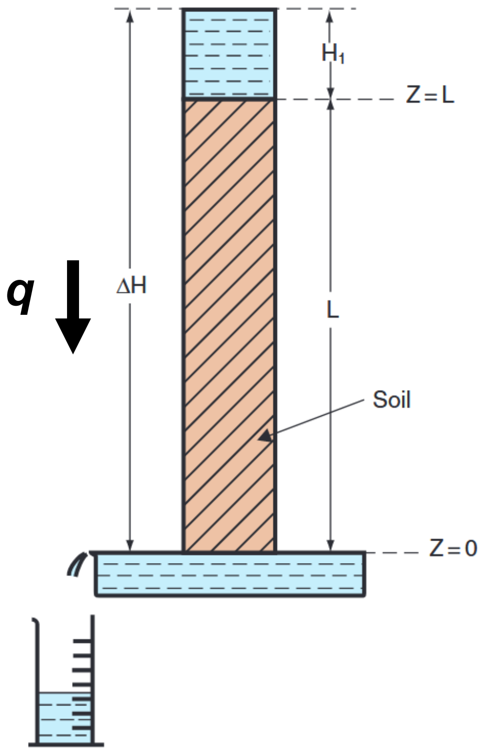


Fig. 7.5. Downward flow of water in a vertical saturated column.

Hillel (2003)

Flow in saturated soils (vertical column)

We now examine the case of upward flow in a vertical column, as shown in Fig. 7.6. In this case, the direction of flow is opposite to the direction of the gravitational gradient, and the hydraulic gradient becomes

		Pressure head		Gravity head
Hydraulic head at inflow boundary H_i	=	H_1	+	0
Hydraulic head at outflow boundary H_o	=	0	+	L
Hydraulic head difference $\Delta H = H_i - H_o$	=	H_1	−	L

The Darcy equation is thus:

$$q = K(H_1 - L)/L = KH_1/L - K = K\ \Delta H/L$$

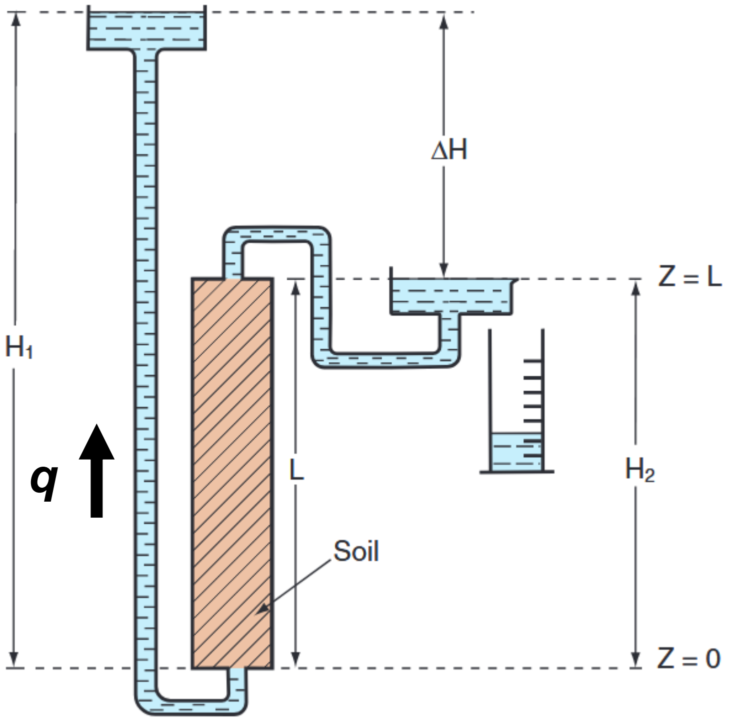


Fig. 7.6. Steady upward flow in a saturated vertical column.

Hillel (2003)

Flow in saturated soils (stratified column)

Figure 7.7 depicts steady flow through a soil column consisting of two distinct layers, each homogeneous within itself and differing from the other layer in thickness and hydraulic conductivity. Layer 1 is at the inlet and layer 2 is at the outlet side of the column. The hydraulic head values at the inlet surface, at the interlayer boundary, and at the outlet end are designated H_1, H_2, and H_3, respectively. At steady flow, the flux through both layers must be equal:

$$q = K_1(H_1 - H_2)/L_1 = K_2(H_2 - H_3)/L_2 \quad (7.10)$$

where q is the flux, K_1 and L_1 are the conductivity and thickness (respectively) of the first layer, and K_2 and L_2 are the same for the second layer. Here we have disregarded any possible contact resistance between the layers. Thus,

$$H_2 = H_1 - qL_1/K_1 \quad \text{and} \quad qL_2/K_2 = H_2 - H_3$$

Therefore,

$$qL_2/K_2 = H_1 - qL_1/K_1 - H_3 \quad \text{and} \quad q = (H_1 - H_3)/(L_2/K_2 + L_1/K_1) \quad (7.11)$$

The reciprocal of the conductivity has been called the *hydraulic resistivity*, and the ratio of the thickness to the conductivity ($R_h = L/K$) has been called the hydraulic resistance per unit area. Hence,

$$q = \Delta H/(R_{h1} + R_{h2}) \quad (7.12)$$

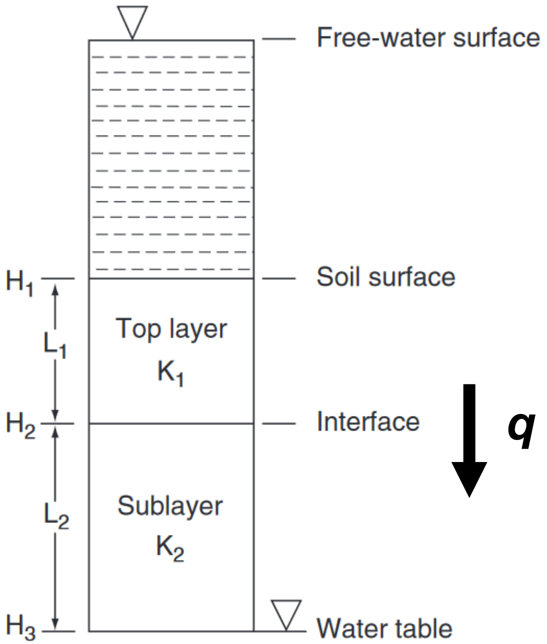


Fig. 7.7. Downward flow through a composite column.

Hillel (2003)

Flow in saturated soils (stratified column)

2. *Ecoulement vertical dans un sol stratifié*

Soit une colonne de sol saturé constituée de 2 couches superposées. La couche supérieure d'épaisseur L_1 présente une conductivité hydraulique K_1, la couche inférieure d'épaisseur L_2 une conductivité hydraulique K_2. Le système est alimenté sous une charge constante l. Calculer le flux en régime permanent.

Soient H_e la valeur de la charge hydraulique à l'entrée, H_i à l'interface entre les deux couches et H_s à la sortie.

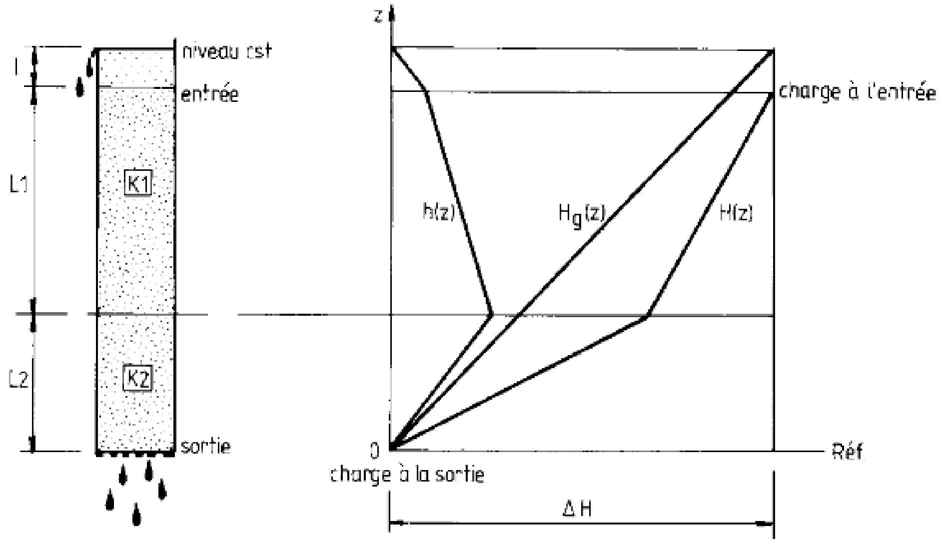


Fig. 5 : Diagramme des potentiels dans un milieu saturé à 2 couches

See Notes 3 and Excel file in Moodle

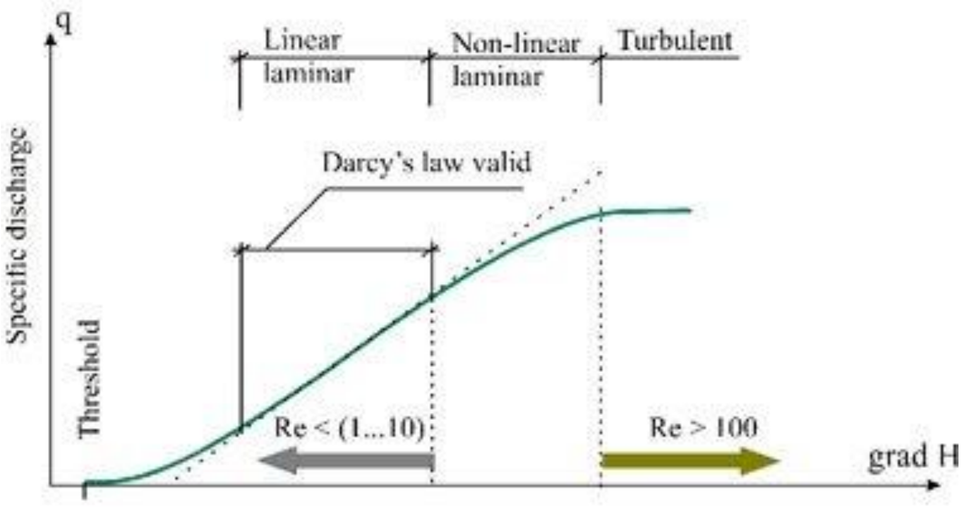
Self-Study

Limitations of Darcy’s law

Darcy' law – which supposes a laminar flow – is valid for **Reynolds number** (i.e., ratio of inertial forces to viscous forces) less than 1, but the upper limit can be extended up to 10. The inception of the turbulent flow can be located at Reynolds numbers greater than 60…100. Between the laminar and the turbulent flow there is a transition zone, where the flow is laminar but non-linear. In a general way, Darcy's law can be written:

$$q = -K\left(\frac{dH}{dx}\right)^m$$

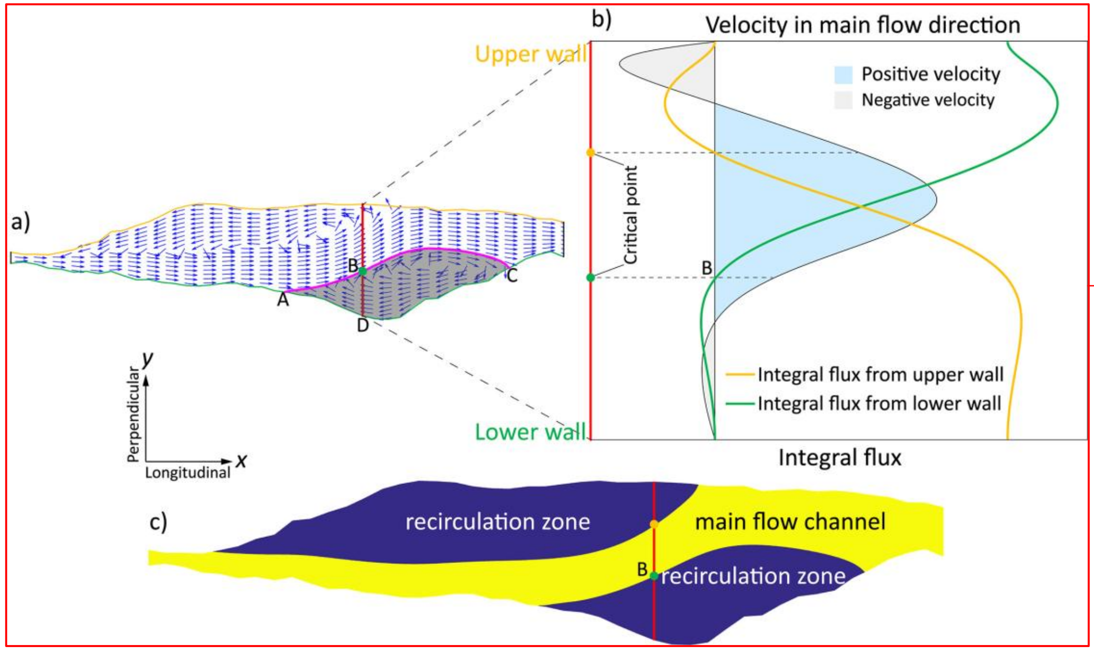
- $m = 1$ mlaminar flow (Darcy is valid)
- $m \neq 1$ in karstic limestones and coarse gravel with Re>100



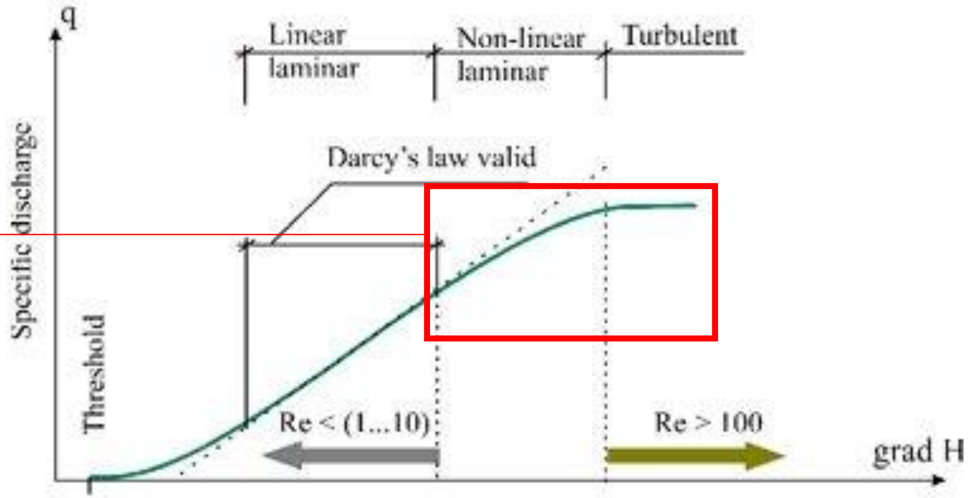
VICAIRE - Module 3 - Chapter 5 (epfl.ch)

Deviations from Darcy’s law may also occur at the opposite end of the flow-velocity range, namely, at **low gradients and in narrow pores**. A possible reason for this anomaly is that the water in close proximity to the particle surfaces and subject to their adsorptive force fields may be more rigid than ordinary bulk water and may exhibit the properties of a non-Newtonian fluid.

Limitations of Darcy's law



Zhou et al. (2018), WRR



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$$-\nabla H = aq + bq^2$$

Darcy

Forchheimer

Vol. 39, No. 4 TRANSACTIONS, AMERICAN GEOPHYSICAL UNION August 1958

On the Theoretical Derivation of Darcy and Forchheimer Formulas

S. IRMAY

$$H = \frac{P}{g\rho_w} + z + \frac{v^2}{2g}$$

Darcy's Law (analogies)

Mathematically, Darcy's law is similar to the linear transport equations of classical physics, such as Ohm's law (stating that the current, or flow rate of electricity, is proportional to the electrical potential gradient), Fourier's law (the rate of heat conduction is proportional to the temperature gradient), and Fick's law (the rate of diffusion is proportional to the concentration gradient).

General Form:

$$\boldsymbol{q} = -K\nabla\phi$$

$\boldsymbol{q}$ = flux (*vector field*)
K = tranfer coefficient
ϕ = potential (*scalar field*)
$\nabla\phi$ = driving force (*gradient*)

Process	Law	$\boldsymbol{q}$	K	ϕ
Water movement (porous media)	Darcy	Water flux	Hydraulic conductiviy	Hydraulic potential, H
Heat tranfer	Fourier	Heat flux	Thermal conductivity	Temperature, T
Solute tansport	Fick	Solute flux	Diffusion coefficient	Concentration, C

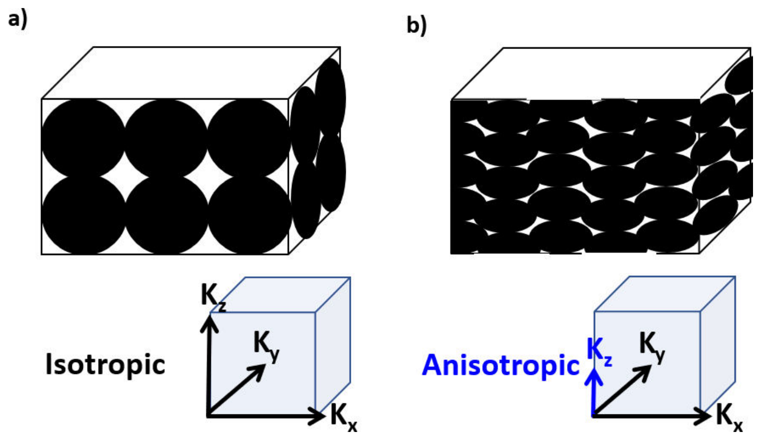
Darcy's Law (generalization)

In short: $\boldsymbol{q} = -K\nabla H = -K\nabla(h+z)$

$$\boldsymbol{q} = (q_x, q_y, q_z) = \left(-K_x\frac{\partial H}{\partial x}, -K_y\frac{\partial H}{\partial y}, -K_z\frac{\partial H}{\partial z}\right)$$

Since: $H = h + z$

$$\boldsymbol{q} = \left(-K_x\frac{\partial h}{\partial x}, -K_y\frac{\partial h}{\partial y}, -K_z\left(\frac{\partial h}{\partial z}+1\right)\right)$$



$K_x = K_y = K_z = K$ $\qquad$ $K_x \neq K_z$

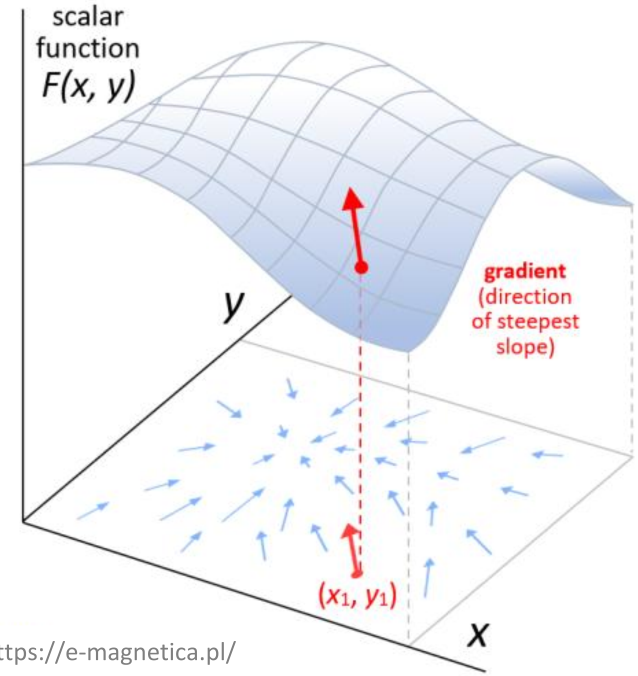
Brief recap: Gradient, divergence, etc

EPFL

$f = f(x, y, z)$ Scalar-valued differentiable function

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ Gradient

$\nabla f = grad(f) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ Gradient of f (vector field)



https://e-magnetica.pl/

$\boldsymbol{F} = (F_x, F_y, F_z)$ vector field

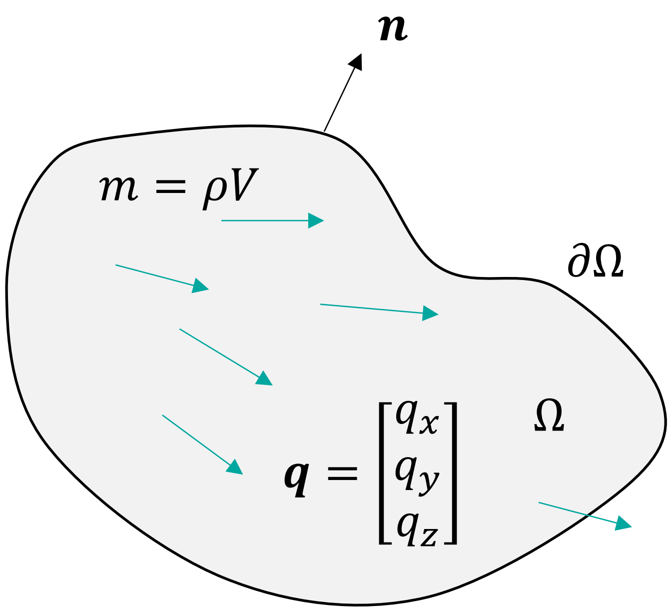
$\nabla \cdot \boldsymbol{F} = div(\boldsymbol{F}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$ Divergence (scalar field)

$div(\nabla f) = \nabla \cdot \nabla f = \nabla^2 f = \Delta f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2}$ Laplacian (scalar field)

Divergence theorem

Gauss (or divergence) Theorem

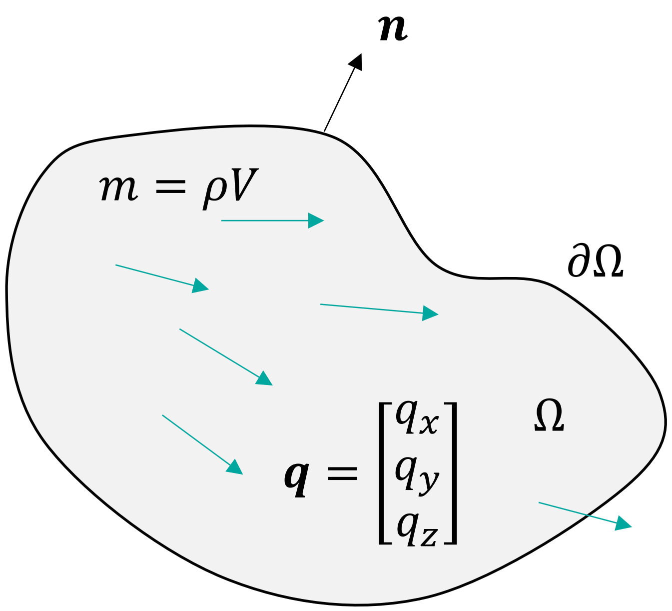
$$\int_{\Omega} (\nabla \cdot \boldsymbol{q})\, dV = \oint_{\partial\Omega} (\boldsymbol{q} \cdot \boldsymbol{n})\, dS$$



The divergence theorem states that any **continuity equation** can be written in a differential form (in terms of a divergence) and an integral form (in terms of a flux)

Continuity equation (Divergence theorem)

$$\begin{Bmatrix} \text{The rate at which} \\ \text{the amount of} \\ m \text{ (e.g. mass) in } \Omega \\ \text{increases} \end{Bmatrix} = -\begin{Bmatrix} \text{The rate at which } m \\ \text{leaves } \Omega \text{ across } \partial\Omega \end{Bmatrix}$$



$$\frac{\partial}{\partial t}\int_{\Omega} \rho dV = -\oint_{\partial\Omega} (\boldsymbol{q}\cdot\boldsymbol{n})\, dS$$

integral form

Ω does not change in time + Divergence theorem

$$\int_{\Omega} \frac{\partial \rho}{\partial t} dV = -\int_{\Omega} (\nabla\cdot\boldsymbol{q})\, dV$$

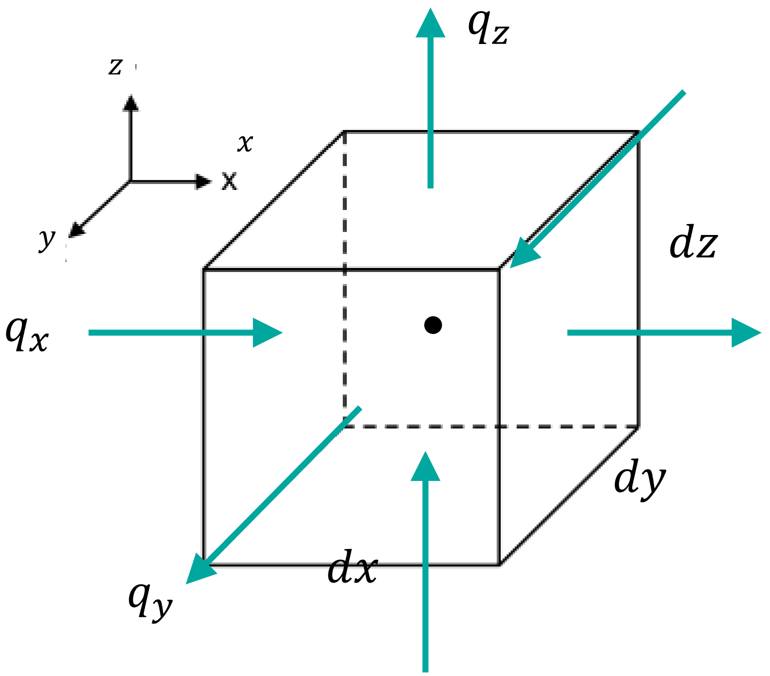
$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla\cdot\boldsymbol{q}}$$

differential form

Continuity equation (Taylor expansion)

$$\frac{\partial m}{\partial t} = IN - OUT$$

With: $m = \rho(x,y,z) \cdot V$



Taylor expansion

$$q_x + \frac{\partial q_x}{\partial x} dx + \frac{1}{2!}\frac{\partial^2 q_x}{\partial x^2} dx^2 + \cdots$$

$$IN = q_x dydz + q_y dxdz + q_z dxdy$$

$$OUT = \left(q_x + \frac{\partial q_x}{\partial x} dx\right) dydz + \left(q_y + \frac{\partial q_y}{\partial y} dy\right) dxdz + \left(q_z + \frac{\partial q_z}{\partial z} dz\right) dxdy$$

$$\frac{\partial m}{\partial t} = \frac{\partial \rho}{\partial t} dxdydz = IN - OUT = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) dxdydz$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{q}$$

Continuity equation

The conservation (or continuity) equation states that the temporal variation of the variable considered (water content, density of heat, concentration, etc.) is equal to the spatial variation of the flow, corrected for possible contributions, losses or transformations within the system

General Form:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{q} + \sum_i r_i$$

ρ = Volumetric concentration of the variable considered

q = flux across the system boundaries

r_i = rate of production, degradation, or transfromation within the domain (many sources/sinks *i* may exist)

Sustance	ρ	Unit	q	Unit
Water	Water content, θ	$m^3\ m^{-3}$	Darcy's law	$m\ s^{-1}$
Heat	Quantity of heat	$J\ m^{-3}$	Fourier's law	$J\ m^{-2}\ s^{-1}$
Chemical substance or gas	Concentration	$kg\ m^{-3}$	Fick's law	$kg\ m^{-2}\ s^{-1}$

Saturated flow

Darcy's law, by itself, is sufficient only to describe steady, or stationary, flow processes, in which the flux remains constant and equal throughout the conducting medium (and hence the potential and gradient at each point remain constant in time). Unsteady, or transient, flow processes, in which the magnitude and possibly even the direction of the flux and potential gradient vary in time, require also the law of conservation of matter:

$$\frac{\partial \theta}{\partial t} = -\nabla \cdot \boldsymbol{q}$$

Conservation of mass

Darcy's law →

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (K \nabla H)$$

General Flow equation

Gravitational + pressure head →

$$\frac{\partial \theta}{\partial t} = \nabla \cdot [K(\nabla h + \nabla z)]$$

In 1D:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[K_x\left(\frac{\partial h}{\partial x}\right)\right]$$

horizontal flow ($\partial z/\partial x = 0$)

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[K_z\left(\frac{\partial h}{\partial z} + 1\right)\right]$$

vertical flow ($\partial z/\partial z = 1$)

If **saturated soil** with incompressible matrix ($\partial \theta/\partial t = 0$), isotropic ($K_x = K_y = K_z$) and homogeneous soil ($K$ is the same everywhere) we get:

$$\nabla^2 H = 0$$

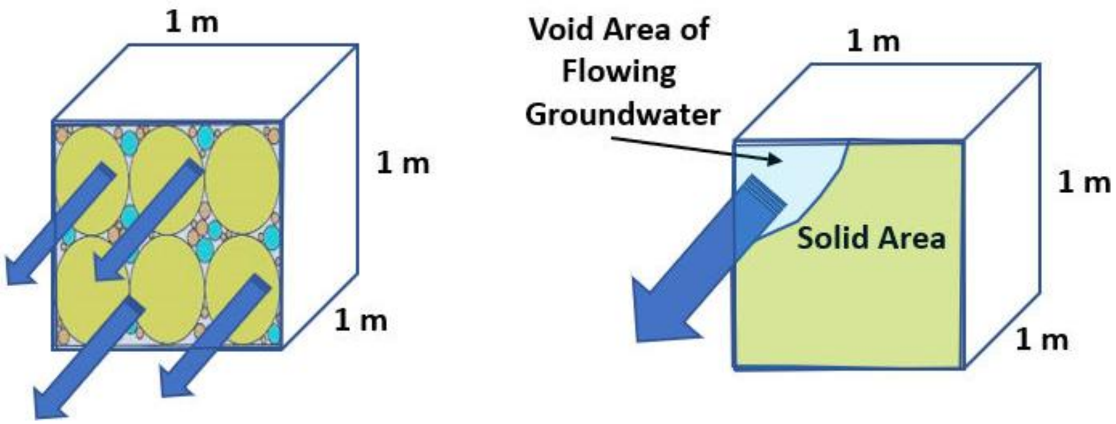
Laplace Equation (elliptic PDE)

Flux, flow velocity, and tortuosity

Flux = the volume of water V passing through a unit cross-sectional area A (perpendicular to the flow direction) per unit time t [L T^{-1}]

The flux has dimensions of velocity, yet the actual flow velocity is something different. Specifically:

- one cannot refer to a single velocity of liquid flow, but at best to an average velocity;
- flow does not take place through the entire cross-sectional area A, and the average velocity of the liquid must be greater than the flux q because the "active" area is smaller than A



The Groundwater Project

$$q = \frac{Q}{A} \qquad v = \frac{Q}{An_e} = \frac{q}{n_e}$$

Effective porosity (n_e)

Note: if $n_e = \frac{1}{3}$, then $v = 3q$

Flux, flow velocity, and tortuosity

Flux = the volume of water V passing through a unit cross-sectional area A (perpendicular to the flow direction) per unit time t [L T^{-1}]

The flux has dimensions of velocity, yet the actual flow velocity is something different. Specifically:

- one cannot refer to a single velocity of liquid flow, but at best to an average velocity;
- flow does not take place through the entire cross-sectional area A, and the average velocity of the liquid must be greater than the flux q because the "active" area is smaller than A

Tortuosity, *T* = the ratio of the average roundabout path to the apparent, or straight, flow path; that is, the ratio of the average length of the pore passages

$$T = \frac{I}{L} \geq 1 \longrightarrow \textbf{Tortuosity factor:} \quad \tau = \frac{1}{T}$$

$$0.3 < \tau < 0.7$$

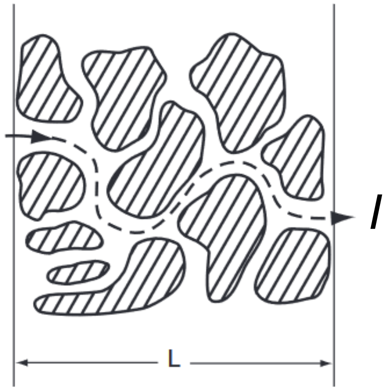


Fig. 7.8. Flow path tortuosity in the soil. Hillel (2003)

Hydraulic conductivity (permeability & fluidity)

The hydraulic conductivity K is not a property of the soil alone. Rather, it depends jointly on the attributes of the soil (porosity, tortuosity, pore size, …) and of the fluid (density, viscosity).

$$K = kf \quad [L\,T^{-1}] \quad \Rightarrow \quad K = \frac{k\rho g}{\eta}$$

Where: $k =$ Intrinsic permeability $[L^2]$

$f = \frac{\rho g}{\eta}$ Fluidity $[L^{-1}T^{-1}]$

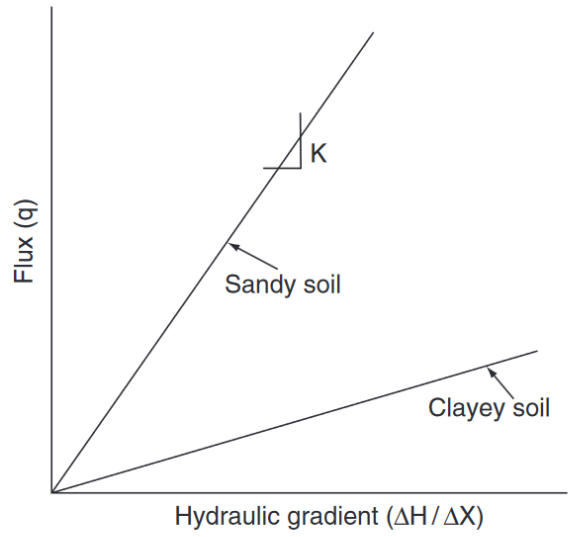


Fig. 7.9. The linear dependence of flux on hydraulic gradient, the hydraulic conductivity being the slope (i.e., the flux per unit gradient).

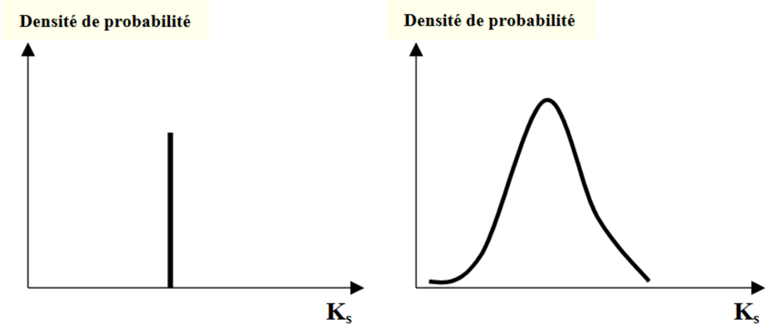
Hillel (2003)

Unconsolidated deposits	Hydraulic conductivity (m/s)
Dense clay	10^{-13} ……10^{-8}
Weathered clay	10^{-8} ……10^{-6}
Silt	10^{-7} ……10^{-5}
Alluvial deposits	10^{-5} ……10^{-3}
Fine sand	10^{-5} ……10^{-4}
Medium sand	$5x10^{-4}$ ……$5x10^{-3}$
Coarse sand	10^{-4} ……10^{-3}
Fine gravel	10^{-3} ……$5x10^{-1}$
Medium gravel	$5x10^{-2}$ ……10^{-1}
Coarse gravel	10^{-2} ……$5x10^{-1}$

https://echo2.epfl.ch/

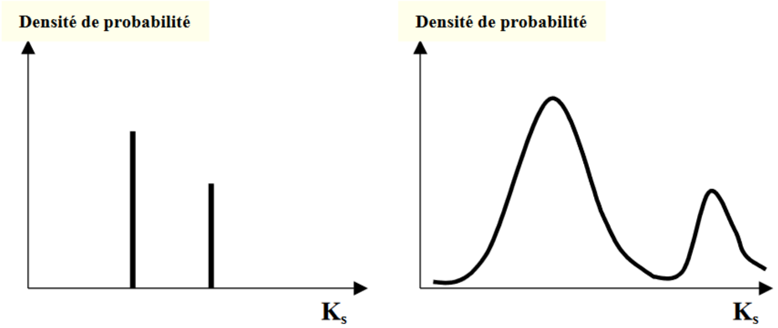
Hydraulic conductivity (spatial variability)

Milieu homogène (FDP* unimodale)



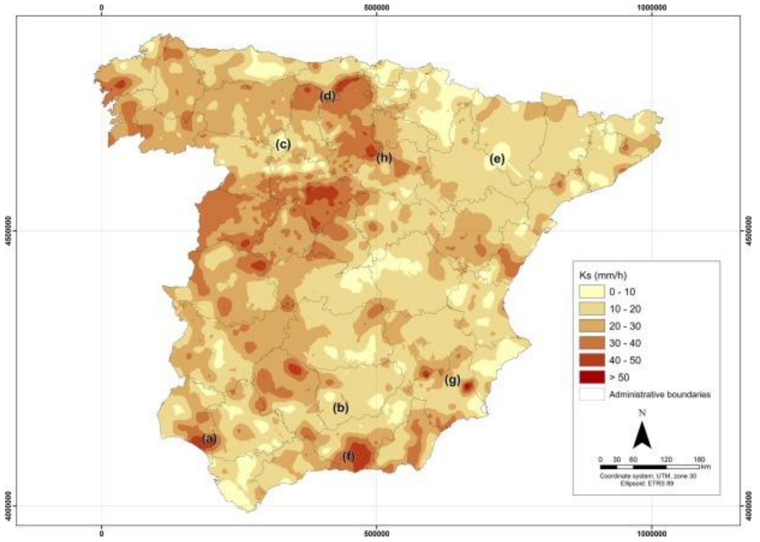
Uniforme **Non uniforme**

Milieu hétérogène (FDP plurimodale)



*** FDP: fonction de densité de probabilité**

Saturated Hydraulic Conductivity



Ferrer-Julia et al. (2021)

Estimating the Saturated Hydraulic Conductivity

In the absence of direct measurements, **empirical formulas** can be used to link saturated hydraulic conductivity (K_s, cm/s) to certain soil properties (e.g., granulometry, porosity, pore distribution, specific surface):

Hazen formula (for sands):

$$K_s = (D_{10})^2$$

Where D_{10} = diameter of the 10 percentile grain size of the material (mm)

Kozeny formula:

$$K_s = 7.94^{\frac{n^3}{(n-1)^2}} \tau d_e^2$$

Where n = porosity
τ = temperature correction coefficient
d_e = effective diameter

Measuring Saturated Hydraulic Conductivity

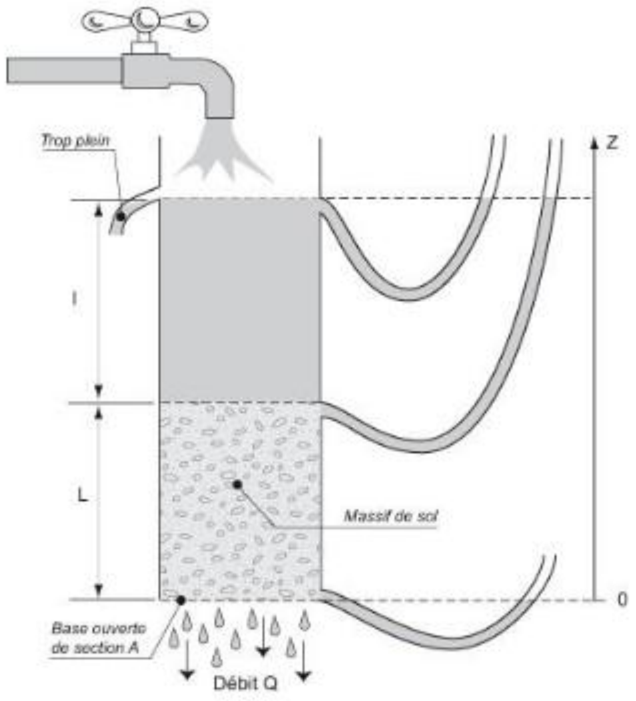
Experiments may be designed to use Darcy's law for determining the saturated hydraulic conductivity (K_s) from measurements of hydraulic heads and water flux density (denoted as flux) from a soil column of a known geometry. We can use either the vertical or the horizontal setup.

- The **constant head method** is based on maintaining constant heads across the soil sample

- The **falling head method**, conducted by recording the initial and final depths of the ponded water, expressed as pressure head in length units, by means of a *falling head permeameter*.

Measuring Saturated Hydraulic Conductivity

Constant head method



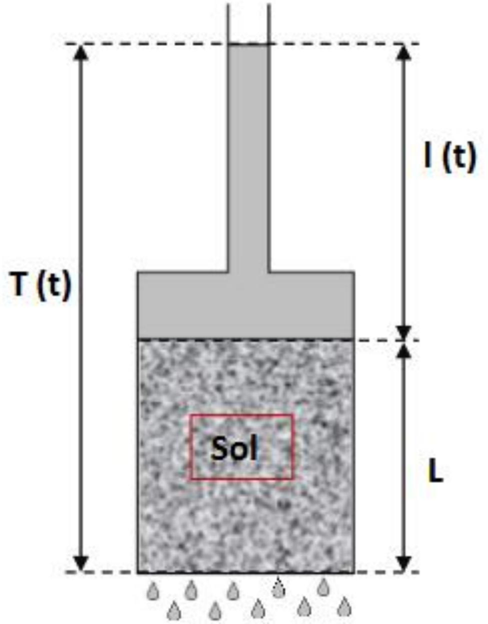
$$Q = -K_s \frac{I + L}{L} S$$

$$K_s = -\frac{QL}{S(I + L)}$$

Note: flow Q is negative because it is oriented in the opposite direction of the axis z

Measuring Saturated Hydraulic Conductivity

Falling head method



$$Q = \frac{dV}{dt} = S\frac{dT(t)}{dt}$$

$$Q = -K_s\frac{L + I(t) - 0}{L}S = -K_s\frac{T(t)}{L}S$$

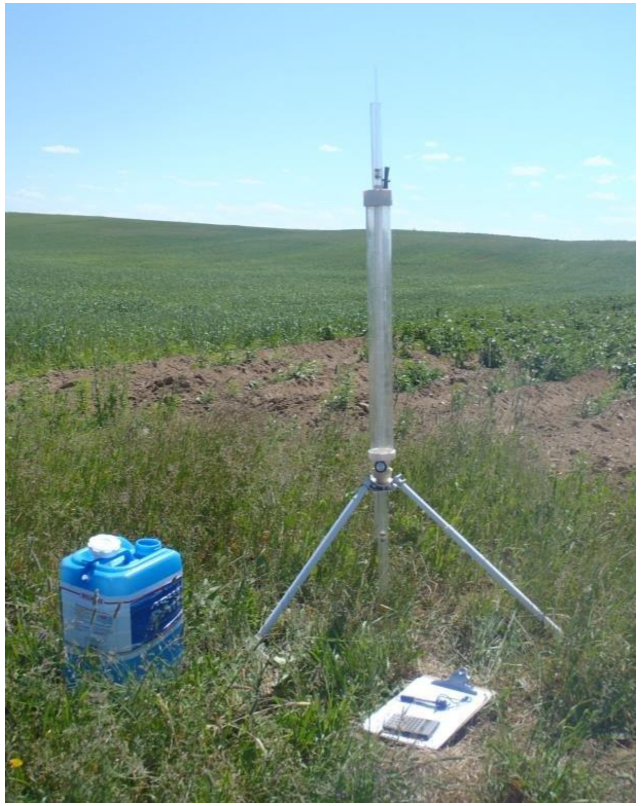
Upon integration

$$-K_s\frac{S}{L}t = S \cdot ln\left(\frac{T(t)}{T_0}\right) = -S \cdot ln\left(\frac{T_0}{T(t)}\right)$$

$$K_s = \frac{SL}{St} \cdot ln\left(\frac{T_0}{T(t)}\right)$$

Measuring Saturated Hydraulic Conductivity

Field measurements



Guelph permeameter

Double-ring infiltrometer

Amoozemeter

Useful links

- **Flow Equations in Porous Media:** see here
- **Derivation of Darcy's law (from Navier-Stokes equation):**
  - Hubbert (1957)
  - Neuman (1977)
- **Analogies** (Darcy, Ohm, Fourier, and Fick laws):
  - Soil water and Heat
  - Pfister (2014) – see Appendix C

This week exercises & assignments

- **Exercises** for **Weeks 4** are available in Moodle
  - In case you want to try more problems, additional exercises are also available (just try the problem before checking the solution!)
- **Computer Lab:** 1D infiltration model (Assignment 3)
- **For next week:** Read pg. 14-27 of Notes 3.pdf

Appendix: Conservation of mass (divergence theorem)

EPFL

We now formulate the law of conservation in integral form for the physical quantity Q. Let $D \subset \mathbb{R}^3$ be the domain of ψ and $\mathbf{j}$, and let Ω be an *arbitrary* bounded subset, as in the statement of Gauss' theorem. If the quantity Q is neither created nor destroyed in D (i.e. no sources or sinks), then conservation of Q has the following form:

$$\left\{ \begin{array}{c} \text{the rate at which the} \\ \text{amount of } Q \text{ in } \Omega \\ \text{increases} \end{array} \right\} = - \left\{ \begin{array}{c} \text{the rate at which } Q \\ \text{leaves } \Omega \text{ across } \partial\Omega \end{array} \right\}.$$

The minus sign is needed because the amount of Q in Ω *decreases* if the flux across $\partial\Omega$ is *positive*.

$$\frac{d}{dt} \iiint_{\Omega} \psi dV = - \iint_{\partial\Omega} \mathbf{j} \cdot \mathbf{n}\, dS.$$

Since ψ is of class C^1 and Ω does not change with time,

$$\frac{d}{dt} \iiint_{\Omega} \psi dV = \iiint_{\Omega} \frac{\partial \psi}{\partial t} dV.$$

In addition we can use Gauss' theorem to write the surface integral as a triple integral,

$$\iint_{\partial\Omega} \mathbf{j} \cdot \mathbf{n}\, dS = \iiint_{\Omega} \nabla \cdot \mathbf{j}\, dV.$$

$$\iiint_{\Omega} \left(\frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{j} \right) dV = 0.$$

$$\frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{j} = 0$$